Prop (Clairaut's Theorem): if $f(x,y)$ has cts mixed second order partial derivatives on an open disk, then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ on the disk

Notation: $f_x = \frac{\partial f}{\partial x}$ $\quad f_y = \frac{\partial f}{\partial y}$

$f_{xx} = (f_x)_x = \frac{\partial^2 f}{(\partial x)^2}$

$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}[f]\right) = \frac{\partial^2 f}{\partial y \partial x}$

↑

professor prefer this notation in the proof

Pf: Let f have cts second-order mixed partials on an open disk $D \subseteq \mathbb{R}^2$ and suppose $(a,b) \in D$

Consider $\Delta(h) = (f(a+h, b+h) - f(a+h, b)) - (f(a, b+h) - f(a,b))$

Define $\alpha(x) = f(x, b+h) - f(x, b)$ and

notice $\quad \alpha(a+h) - \alpha(a) = f(a+h, b+h) - f(a+h, b)$

$h = (a+h) - a \quad\quad - (f(a, b+h) - f(a,b))$

$[a, a+h] \quad\quad = \Delta(h)$

for all $h \neq 0$ where $(a+h, b)$

$(a+h, b+h)$ $(a, b+h) \in D$

By mean value theorem, for ~~early~~ every given h

Come from MVT $\rightarrow$

There is C_h with $|a - C_h| \le |h|$ so that

$h\,\alpha'(C_h) = \alpha(a+h) - \alpha(a) = h(f_x(C_h, b+h) - f_x(C_h, b))$

$\therefore \Delta(h) = \alpha(a+h) - \alpha(a) = h(f_x(C_h, b+h) - f_x(C_h, b))$ $*$

Next apply MVT to $\beta(y) = f_x(C_h, y)$ to obtain a d_h with $|b - d_h| \le |h|$ so that

$$h\,\beta'(d_h) = \beta(b+h) - \beta(b)$$
$$= f_x(C_h, b+h) - f_x(C_h, b)$$

$\therefore$ substituting into $*$ yields

$$\Delta(h) = h(f_x(C_h, b+h) - f_x(C_h, b))$$
$$= h(h\beta'(d_h))$$
$$= h^2 (f_x)_y (C_h, d_h)$$
$$= h^2 f_{xy} (C_h, d_h)$$

we may ~~reat~~ repeat the argument to obtain (γ_h, δ_h) for all h such that

$|a - \gamma_h| \le |h|$

$|b - \delta_h| \le |h|$

and $\Delta(h) = h^2 f_{yx}(\gamma_h, \delta_h)$

Notice = by construction that $\lim_{h\to 0} (C_h, d_h) = (a, b)$

$= \lim_{h\to 0} (\gamma_h, \delta_h)$

Finally we have

$$f_{xy}(a,b) = f_{xy}\left(\lim_{h\to 0}(c_h, d_h)\right)$$

$$\overset{\text{continuity}}{=} \lim_{h\to 0} f_{xy}(c_h, d_h)$$

$$\overset{\text{computed the equality}}{=} \lim_{h\to 0} \frac{\Delta h}{h^2}$$

$$= \lim_{h\to 0} f_{yx}(\gamma_h, \delta_h)$$

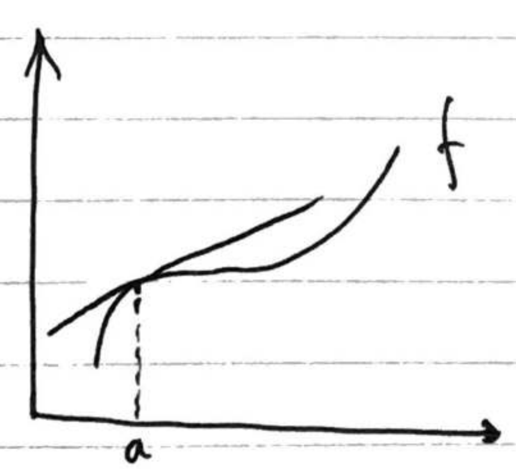
$$\overset{\text{continuity}}{=} f_{yx}\left(\lim_{h\to 0}(\gamma_h, \delta_h)\right)$$

$$= f_{yx}(a,b)$$

hence, we're proved the result

§ 14.? · Linear Approximation of Multivariable Functions

Idea: In ~~of~~ Calculus I, near a point on graph(f), f is "approximated well" by the tangent line



as $x \to a$, the error approximating f by f's tangent line going to 0.

In cal III, we approximate graph(f) near a point by tangent $\underbrace{\text{(hyper)}}$ plane ~~ab~~

↳ Ignore unless we had more than 2 variables

In 2-variable we got a tangent plane

In cal I, the tangent line has formula

$$y - f(a) = f'(a)(x-a)$$

For a function $f(x,y)$ to approximate f near (a,b) we got formula:

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

hence, the linear approximation to f at (a,b) is $z = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$

Ex: Find an equation of the tangent plane to $f(x,y) = x^2 + xy - y^2$ at $(1,2)$

Sol: Using the formula $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$f_x = 2x + y \qquad f_x(1,2) = 2\cdot 1 + 2 = 4$

$f_y = x - 2y \qquad f_y(1,2) = 1 - 2\times 2 = -3$

$f(1,2) = 1^2 + 1\cdot 2 - 2^2 = -1$

hence the tangent plane is $z = -1 + 4(x-1) - 3(y-2)$

$= 4x - 3y + 2$

Ex: Compute the tangent plane to $f(x,y) = \ln(x-2y)$ at $(3,1,0)$

Sol: we need to compute the tangent plane
$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
$f(a,b) = 0$

$f_x = \frac{1}{x-2y}$ $f_x(3,1) = \frac{1}{3-2\cdot 1} = 1$
$f_y = \frac{-2}{x-2y}$ $f_y(3,1) = \frac{-2}{3-2} = -2$

The tangent plane is $z = 0 + 1(x-3) - 2(y-1)$
$= x - 2y - 5$

Definition: Let f be a function of variables $(x_1, x_2, x_3 \cdots x_n)$ the total differential of f is $df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \cdots \frac{\partial f}{\partial x_n} dx_n$

symbols representing change in corresponding variables ....

Ex compute the total differential of $f(x,y,z) = e^x y^2 (z-5)^{\frac{1}{2}}$

Sol: $f_x(x,y,z) = e^x y^2 (z-5)^{\frac{1}{2}}$
$f_y(x,y,z) = 2e^x y (z-5)^{\frac{1}{2}}$
$f_z(x,y,z) = \frac{1}{2} e^x y^2 (z-5)^{-\frac{1}{2}}$

$\therefore\ df = f_x dx + f_y dy + f_z dz$

$= e^x y^2 (z-5)^{\frac{1}{2}} dx + 2e^x y (z-5)^{\frac{1}{2}} dy + \frac{1}{2} e^x y^2 (z-5)^{-\frac{1}{2}} dz$

Ex : Estimate Δf from $(1, 1, 6)$ to $(1.5, 1.5, 5.5)$

Sol : $\Delta f \approx df$ where $dx_i \approx \Delta x_i$

So $\Delta f \approx f_x(1,1,6)\Delta x + f_y(1,1,6)\Delta y + f_z(1,1,6)\Delta z$

$= e(1.5-1) + 2e(1.5-1) + \frac{1}{2}e(5.5-6)$

$= e(\frac{1}{2} + 2\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}) = e\cdot\frac{5}{4}$